Regularity of paths of the Wiener process and of the Brownian sheet

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Abstract

Paul Lévy showed in 1937, that the sample paths of the Wiener process $(W_t)_{0\le t\le 1}$ lie almost surely in the Hölder space $\mathcal{C}^g([0,1])$, where

$$g(t)=\sqrt{|t|\cdot\log\frac{1}{|t|}}$$

for $|t|>0$ small. In the language of modern function spaces, this space is identified with the Besov space of logarithmic smoothness $B^{1/2,1/2}_{\infty,\infty}([0,1])$. Later on, Ciesielski showed, that these paths can also be almost surely found in $B^{1/2}_{p,\infty}([0,1])$ for every $1\le p<\infty$ and in the Besov-Orlicz space $B^{1/2}_{\Phi_2,\infty}([0,1])$, where $\Phi_2(t)=\exp(t^2)-1$ for $t>0$.

We show that all these results can be rather easily recovered from the Lévy's decomposition of the Wiener process into the Faber system of shifted and dilated hat functions. Moreover, we define also new (and strictly smaller) function spaces, where the paths of the Wiener process lie almost surely.

We also recall the work of Kamont, who generalized this approach to the bivariate random field $(B(s,t))_{0\le s,t\le 1}$ called Brownian sheet leading naturally to function spaces of dominating mixed smoothness. Also in this case, we recover the known results and propose new (and strictly smaller) function spaces, which contain almost all its paths.